Quantum line bundles via Cayley-Hamilton identity

2001 J. Phys. A: Math. Gen. 34 4553

(http://iopscience.iop.org/0305-4470/34/21/309)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 4553–4569 www.iop.org/Journals/ja PII: S0305-4470(01)21622-9

Quantum line bundles via Cayley–Hamilton identity

D Gurevich[1] **and P Saponov**[2]

[1] ISTV, Université de Valenciennes, 59304 Valenciennes, France
[2] Theory Division, Institute for High Energy Physics, 142284 Protvino, Russia

Received 30 January 2001

Abstract

As shown by Pyatov and Saponov (1995 *J. Phys. A: Math. Gen.* **28** 4415–21) and Gurevich *et al* (1997 *Lett. Math. Phys.* **41** 255–64), the matrix $L = ||l_i^j||$, whose entries l_i^j are generators of the so-called reflection equation algebra (REA), is subject to some polynomial identity resembling the Cayley–Hamilton identity for a numerical matrix. Here a similar statement is presented for a matrix whose entries are generators of a filtered algebra that is a 'non-commutative analogue' of the REA. In an appropriate limit we obtain a similar statement for the matrix formed by the generators of the algebra $U(gl(n))$. This property is used to introduce the notion of line bundles over quantum orbits in the spirit of the Serre–Swan approach. The quantum orbits in question are presented explicitly as some quotients of one of the algebras mentioned above both in the quasiclassical case (i.e. that related to the quantum group $U_q(sl(n))$) and a non-quasiclassical one (i.e. that arising from a Hecke symmetry with non-standard Poincaré series of the corresponding symmetric and skew-symmetric algebras).

PACS numbers: 0210W, 0220, 0240

Mathematics Subject Classification: 17B37, 81R50

1. Introduction

Let $G = GL(n)$, $\boldsymbol{g} = \text{Lie}\,(G)$ over the field[3] $k = \boldsymbol{R}$ or $k = \boldsymbol{C}$. As usual, let us identify $\boldsymbol{g}$ and $\boldsymbol{g}^*$ and consider a matrix $A \in \boldsymbol{g}^*$ with pairwise distinct eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$. Denote by M_μ, $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ a G-orbit of the matrix A w.r.t. coadjoint action of $\boldsymbol{g}$. Then the orbit M_μ being an affine algebraic variety is defined by the following system of equations:

$$\text{Tr}\; A^k = c_k = \sum \mu_i^k \qquad k = 1, 2, 3, \ldots, n. \tag{1.1}$$

Let $k(M_\mu)$ be its coordinate ring.

[3] The choice of the field is similar to that in the classical case. If $k = \boldsymbol{R}$ the entries of the quantum R-matrix R are assumed to be real.

A typical example of a line bundle over M_μ is an eigenspace corresponding to an eigenvalue μ_i, i.e. the space of the vectors $v \in V^*$, V^* being the right $\boldsymbol{g}^*$-module, such that

$$v\,A = \mu_i\, v \qquad v \in V^*. \tag{1.2}$$

Then this line bundle itself is an algebraic variety: it is defined in the space $\boldsymbol{g}^* \times V^*$ by system (1.1), (1.2). This variety (i.e. the total space of the line bundle in question) will be denoted E_{μ_i}. The coordinate ring of this variety $k(E_{\mu_i})$ has the structure of a $k(M_\mu)$-module.

This example is a particular case of a one-to-one correspondence between an algebraic vector bundle over an affine algebraic variety and finitely generated projective modules over its coordinate ring realized in [Se] (a similar correspondence on compact smooth varieties was established in [Sw]).

Let us remark that the projectivity of the $k(M_\mu)$-module $k(E_{\mu_i})$ can be shown by means of the projector

$$P_i = \prod_{j\neq i}^{n} \frac{A - \mu_j\,\mathrm{id}}{\mu_i - \mu_j} \tag{1.3}$$

since this $k(M_\mu)$-module can be identified with Im P_i.

Our main purpose is to generalize the construction of considered (and some 'derived') line bundles to the quantum case.

To our knowledge the first attempt to realize a line bundle over a quantum sphere in terms of projective modules was featured in [HM]. A quantum analogue of projector (1.3) over a quantum sphere (or, which is the same, a quantum hyperboloid if we ignore the involution operation) was constructed there.

In this paper we suggest a regular way of constructing projective modules for a generic μ, which are the quantum deformations of $k(M_\mu)$-modules $k(E_{\mu_i})$. More precisely, we construct quantum two-parameter deformation of the algebra $k(M_\mu)$ and line bundles $k(E_{\mu_i})$ (see below).

The basic question arising from the very beginning is what are quantum analogues of the orbits in $\boldsymbol{g}^*$. A habitual way to introduce such quantum objects makes use of the so-called Hopf–Galois extension (cf [Sh, HM]). This approach allows one to generalize the notion of an ordinary orbit in terms of a couple of Hopf algebras. The famous 'RTT' algebra and some of its Hopf subalgebras are usually employed as such couples. However, such an approach does not enable one either to control the flatness of deformation[4] in the quasiclassical case or to generalize construction of quantum orbits to a non-quasiclassical case.

Let us note that we use the term 'quasiclassical' for objects arising from deformations of classical ones. In this paper all quasiclassical objects in question are endowed with the structure of a $U_q(sl(n))$-module (and as usual the products in all quantum algebras involved are supposed to be $U_q(sl(n))$-covariant). By 'non-quasiclassical' objects we mean those arising from the solutions R of the quantum Yang–Baxter (YB) equation (2.1), whose symmetric and skew-symmetric algebras have non-quasiclassical Poincaré series. These algebras are well defined if we assume R to be a Hecke symmetry (i.e. a solution of the YB equation subject to the Hecke condition (2.2)). For details the reader is referred to [G], where a large family of non-quasiclassical Hecke symmetries was constructed.

We suggest another way of introducing quantum orbits. The central role in our approach is played by the so-called reflection equation algebra (REA) $\mathcal{L}_q(R)$ (see section 2 for a definition), which can be associated with any (quasiclassical or not) Hecke symmetry R. If R is a

[4] We refer the reader to [DGK] for the rigorous definition of this notion. Roughly speaking, this means that the supply of elements does not change under deformation of the initial object. Let us remark that under a flat deformation A_h of a commutative associative algebra $A = A_0$ the skew-symmetrized term linear in h of the deformed product is a Poisson bracket.

quasiclassical Hecke symmetry then the algebra $\mathcal{L}_q(R)$ (similarly to the RTT algebra) is a flat deformation of the coordinate ring $k(\mathrm{Mat}(n,\ k))$. However, its properties differ drastically from those of the RTT algebra.

The main difference is that the RE algebra possesses a large centre Z. In particular, the quantum trace Tr_q, well defined in this algebra, belongs to Z. On quotienting the RE algebra over the ideal generated by Tr_q we obtain an algebra with $n^2 - 1$ generators, which we consider as a q-analogue of the algebra $k(sl(n)^*) = \mathrm{Sym}(sl(n))$. If instead of the ideal mentioned we take that generated by the elements

$$z - \chi(z) \qquad z \in Z \tag{1.4}$$

where χ is a generic character of Z, we obtain the quotient algebra (denoted $k(M^q_\mu)$), which can be considered as a quantum analogue of semisimple orbits above (we call semisimple orbits those of semisimple elements). Thus, in both the quasiclassical and non-quasiclassical cases such a type of quantum orbit is defined by means of some 'quantum (or braided) algebraic equations' in the spirit of affine algebraic geometry. By abusing the language we call quantum orbits the corresponding 'coordinate rings' in both cases.

Another important difference between the RTT and RE algebras consists in the fact that the latter (being a quadratic algebra) admits a further flat deformation, giving rise to a quadratic–linear algebra which resembles the enveloping algebra $U(gl(n))$. The final object of such a deformation is an algebra $\mathcal{L}_{q,\hbar}(R)$ (see section 2) depending on two parameters, which tends to the RE algebra as $\hbar \to 0$ and to $U(gl(n)_\hbar)$ as $q \to 1$, where the defining relations of $U(gl(n)_\hbar)$ are as follows:

$$a^{i_1}_{j_1} a^{i_2}_{j_2} - a^{i_2}_{j_2} a^{i_1}_{j_1} = \hbar(\delta^{i_2}_{j_1} a^{i_1}_{j_2} - \delta^{i_1}_{j_2} a^{i_2}_{j_1}). \tag{1.5}$$

Hereafter we use the notation $U(\boldsymbol{g}_\hbar)$ for the enveloping algebra of a Lie algebra $\boldsymbol{g}_\hbar$ which differs from $\boldsymbol{g}$ by the factor $\hbar$ introduced into the Lie bracket. We prefer to use the Lie algebra $\boldsymbol{g}_\hbar$ instead of $\boldsymbol{g}$ in order to represent its enveloping algebra $U(\boldsymbol{g}_\hbar)$ as a deformation of the algebra $k(\boldsymbol{g}^*) = \mathrm{Sym}(\boldsymbol{g})$. In a similar way we treat quotients of the algebra $U(\boldsymbol{g}_\hbar)$ as deformations of the corresponding orbits[5].

Similarly to $\mathcal{L}_q(R)$ the algebra $\mathcal{L}_{q,\hbar}(R)$ has a large centre. On quotienting the algebra $\mathcal{L}_{q,\hbar}(R)$ over an ideal resembling (1.4) we obtain a 'quantum non-commutative' analogue of the orbits above. We treat the specialization of this quotient at the point $q = 1$ as a 'classical non-commutative' orbit. Thus, this specialization is just an appropriate quotient of the algebra $U(gl(n)_\hbar)$ (or $U(sl(n)_\hbar)$).

Note that we consider the RE algebra $\mathcal{L}_q(R)$ and its quotients as 'quantum commutative' algebras. Their non-commutative counterparts are the algebra $\mathcal{L}_{q,\hbar}(R)$ and its quotients (all these algebras are well defined in the non-quasiclassical case as well). The term 'classical' means that the product in the algebra in question is G-covariant where G is a usual group. By contrast, 'quantum' means that the product in the algebra in question is covariant w.r.t. a Hopf algebra. In the quasiclassical case this Hopf algebra is just $U_q(sl(n))$. In a non-quasiclassical case an explicit description of a similar Hopf algebra is more complicated (cf [AG] where an attempt to describe such an algebra featured). This is the reason why in a generic case it is more convenient to use the RTT algebra in order to define 'symmetries' of the objects in terms of its coaction (see section 2).

Now let us explain what we understand by quantum analogues of the line bundles above. Replace the matrix A in (1.2) by matrix $L = ||l_i^j||$ subject to (2.4) (this means that the

[5] As for the RTT algebra, it does not have any non-trivial quadratic–linear deformation which could be considered as a q-analogue of $U(gl(n))$ (cf [GR]).

matrix L is formed by the elements l_i^j satisfying the quadratic relations (2.4)). Thus, we have the following system:

$$v_i\, l_j^i - \nu\, v_j = 0 \qquad \nu \in k \tag{1.6}$$

where the summation over repeated indices is assumed. Otherwise stated, we consider the free right $\mathcal{L}_q(R)$-module

$$V \otimes_k \mathcal{L}_q(R) \qquad \text{where} \quad V = \mathrm{span}\,(v_i)$$

and its submodule $\mathcal{R}_\nu$ generated by the lhs of (1.6). Let us restrict ourselves to a 'quantum orbit' $k(M_\mu^q)$. This means that instead of the free $\mathcal{L}_q(R)$-module $V \otimes_k \mathcal{L}_q(R)$ and its submodule $\mathcal{R}_\nu$ we consider the free right $k(M_\mu^q)$-module

$$\mathcal{R}(V,\ M_\mu^q) = V \otimes k(M_\mu^q)$$

and its submodule generated by the lhs of (1.6) (we keep the notation $\mathcal{R}_\nu$ for this).

We call the quotient

$$\mathcal{R}(V,\ M_\mu^q)/\mathcal{R}_\nu$$

if it is non-trivial *a quantum line bundle* over the given quantum orbit $k(M_\mu^q)$. This definition can be extended to a quantum line bundle over 'non-commutative quantum orbits'. For this its suffices to replace 'commutative quantum orbit' in this definition by its 'non-commutative' counterpart (denoted $k(M_\mu^{q\hbar})$). The problem is when the quotient $\mathcal{R}(V,\ M_\mu^q)/\mathcal{R}_\nu$ (or its non-commutative analogue $\mathcal{R}(V,\ M_\mu^{q\hbar})/\mathcal{R}_\nu$) is non-trivial. In the classical commutative case ($q = 1,\ \hbar = 0$) it is so iff ν is a root of the characteristic polynomial of A (i.e. $\nu = \mu_i$ for some i).

In this paper we give a criterion on ν which yields non-triviality of these quotients. This criterion is based on the quantum version of the Cayley–Hamilton (CH) identity for the matrix L found in [GPS]. This version of the CH theorem states that there exists a polynomial P whose coefficients belongs to Z and such that $P(L) = 0$. When we restrict ourselves to a quantum orbit the coefficients of P become numerical. So, we obtain a polynomial $\overline{P}$ with numerical coefficients such that $\overline{P}(L) = 0$. Our main statement says that the quotient module $\mathcal{R}(V,\ M_\mu^q)/\mathcal{R}_\nu$ is non-trivial iff ν is a root of $\overline{P}$ (we assume that the roots μ_i of $\overline{P}$ are pairwise distinct). Moreover, this quotient is projective and in the quasiclassical case it is a flat deformation of its classical counterpart.

Besides, we present here a version of the CH identity valid for the two-parameter algebra $\mathcal{L}_{q,\hbar}(R)$ and by passing to the limit $q \to 1$ we obtain such an identity for the algebra $U(gl(n)_\hbar)$. This allows us to obtain a similar description for 'non-commutative orbits' in both the classical and quantum cases. Let us remark that a version of the CH identity for the algebra $U(gl(n))$ has been known since the late 1960s due to [BL], but in the cited works the identity was established for any finite representation of $U(gl(n))$ and looks like

$$\prod_i (A - \mu_i) = 0$$

where μ_i are *integer* numbers, depending on a given representation. In a sense, our result is more general, since the CH identity is realized with coefficients being elements of the RE algebra itself without using any representation. In particular, this allows us to consider the orbits of general form, where the coefficients μ_i are not obligatory integer numbers. Also a non-commutative version of the CH identity is presented in [G-T]. However it is rather useless for our aims, since the coefficients of the CH polynomial are scalar matrices.

In the classical case besides the above line bundles related to the fundamental vector $sl(n)$-module V (called in the following *basic*), there exist other line bundles which can be obtained via the tensor products of the basic ones. Moreover, the family of all line bundles over a regular algebraic (or a smooth) variety forms a ring w.r.t. the tensor product. Then a natural question arises: what is a regular way to construct quantum line bundles over the 'quantum orbits' which would be different from the basic ones (we will refer to them as *derived* line bundles)? If we want to realize the tensor product of two or more basic line bundles in terms of projective modules we should construct the corresponding projector. In fact the problem of constructing such a projector reduces to the problem of finding the CH identity for the matrix L extended to the tensor product of two (or more) copies of the space V. In the classical case this CH identity can be easily found.

However, it is not so in the quantum case. It is not even clear what is a reasonable way to extend the action of the matrix L to a tensor power of the space V. Remark that any way that is 'reasonable' at first glance leads to an extension of the matrix L for which we are not able to find any polynomial identity which would be a deformation of the classical one (see section 5). Nevertheless, there exists a 'canonical' way to extend the action of the matrix L to the *symmetric* part of $V^{\otimes l}$. Hopefully, for such an extension of the matrix L the CH identity can be found and it is a flat deformation of its classical counterpart. At least, it is so in a particular case when $l = 2$ and rank $(R) = 2$ (this means that the Poincaré series of 'skew-symmetric algebra' of the space V is of the form $P_-(t) = 1 + n\,t + t^2$).

In subsequent publications we will apply our approach to a quantum version of K-theory, which on one hand would enable us to control the flatness of deformation in the quasiclassical case and on the other hand would be valid in the non-quasiclassical case.

The paper is organized as follows. In section 2 we give a description of the RE algebra in comparison with RTT algebra and introduce 'quantum orbits' as some quotients of the former. 'Non-commutative' counterparts of these orbits are introduced as well. In section 3 we present quasiclassical counterparts of these quantum orbits, assuming them to be deformations of generic semisimple ordinary orbits. Section 4 is devoted to construction of 'basic line bundles' over quantum orbits in terms of projective modules. In the last section we discuss a way to define some derived line bundles related to the symmetric product of the basic modules and calculate the corresponding CH identity in the simplest case mentioned above. In the appendix we present calculations of coefficients of the CH identity for 'non-commutative' cases.

2. Reflection equation algebra and quantum orbits

Consider a matrix solution $R^{i_1 i_2}_{j_1 j_2} \in \mathrm{Mat}(n^2, k)$ of the YB equation[6]

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23} \tag{2.1}$$

satisfying the additional Hecke condition

$$R^2 = \mathrm{id} + \lambda R \qquad \text{where} \quad \lambda = q - q^{-1} \tag{2.2}$$

the value of nonzero number $q \in k$ being generic: $q^r \neq 1$ for any integer r. Such solutions will be refered to as Hecke symmetries, and following [G] we will also suppose that the Hecke symmetry R is an even symmetry of finite rank $p \leqslant n$. This means that

$$P_-^{(p+1)} \equiv 0 \qquad \text{and} \qquad \dim\, P_-^{(p)} = 1$$

where $P_-^{(l)}$ stands for the projector of $V^{\otimes l}$ onto its subspace of totally skew-symmetric tensors. It is possible to show that such a Hecke symmetry is closed in the sense of [G]; i.e., the matrix R^{t_1} is invertible. More detailed treatment can be found in [G, GPS].

[6] The standard matrix conventions of [FRT] are used throughout the paper.

With any Hecke symmetry R (quasiclassical or not) we can associate two matrix algebras. One of them, denoted below as $\mathcal{T}_q(R)$ and called the RTT algebra, is generated by n^2 quantities t_j^i, which can be considered as entries of some matrix $T = ||t_j^i||$ subject to the following quadratic relations [FRT]:

$$R_{12}T_1T_2 = T_1T_2R_{12}. \tag{2.3}$$

It is well known that the algebra $\mathcal{T}_q(R)$ possesses the bialgebra structure w.r.t. to the comultiplication

$$\Delta(t_j^i) = t_p^i \otimes t_j^p.$$

In the case of an even Hecke symmetry (quasiclassical or not) and under the assumption that the so-called quantum determinant is central (cf [G]) we can extend this bialgebra structure to the Hopf algebra by introducing the antipodal mapping

$$\mathcal{S} : \mathcal{T}_q(R) \to \mathcal{T}_q(R).$$

Another of the mentioned algebras is so-called RE algebra [7] $\mathcal{L}_q(R)$. The corresponding $n \times n$ matrix $L = ||l_j^i||$ obeys the relation

$$R_{12}L_1R_{12}L_1 = L_1R_{12}L_1R_{12}. \tag{2.4}$$

This algebra can be given a structure of the adjoint comodule w.r.t. the coaction δ of the algebra $\mathcal{T}_q(R)$:

$$\delta(l_j^i) = t_p^i\,\mathcal{S}(t_j^k) \otimes l_k^p. \tag{2.5}$$

Remark 1. As we have said in the introduction we prefer using the RTT algebra as a substitute for the symmetry group since it is well defined in both quasiclassical and non-quasiclassical cases. As for the dual object, its explicit description in a non-quasiclassical case is not easy (cf [AG]). However, in fact we can do without the RTT algebra altogether. Let us also point out that the RE algebra has the structure of a braided Hopf algebra. This property was discovered by Majid (cf [M2]). However, we do not use this property either.

An important feature of both algebras mentioned above is the existence of polynomial identities on the quantum matrices L and T [GPS, IOP], which generalize the well known CH identity of the classical matrix analysis.

For the RE algebra $\mathcal{L}_q(R)$ this identity appears as follows [GPS]:

$$(-L)^p + \sum_{k=0}^{p-1}(-L)^k\sigma_{p-k}(L) \equiv 0. \tag{2.6}$$

Note that the upper limit of summation in the identity is defined by the number $p = \mathrm{rank}(R)$, not by n, the dimension of the space V. The coefficients $\sigma_k(L)$ in the above relation are polynomial combinations of generators l_j^i:

$$\begin{aligned} \sigma_k(L) &= \alpha_k\ \mathrm{Tr}_{(12\ldots p)}\, P_-^{(p)}(L_1R_{12}\ldots R_{k-1,k})^k \\ \alpha_k &= q^{-k(p-k)}[C_p^k]_q \end{aligned} \tag{2.7}$$

where $[C_p^k]_q$ are q-binomial coefficients. Let us note that in the quasiclassical case the set $\{\sigma_k(L)\}$ generates the centre Z of the algebra $\mathcal{L}_q(R)$. We will conjecturally suppose the same property to be true in the non-quasiclassical case as well.

[7] There are known different versions of the RE algebra, cf [KSk, KSa]. We use that introduced in [M1] in terms of braided matrix algebra.

The CH identity for the algebra $\mathcal{T}_q(R)$ is completely different from that above. As was shown in [IOP] the matrix T of generators of $\mathcal{T}_q(R)$ algebra satisfies the following identity:

$$(-T)^{\bar{p}} + \sum_{k=1}^{p-1} (-T)^{\bar{k}} \sigma_{p-k}(T) + \sigma_p(T)\mathcal{D} \equiv 0 \tag{2.8}$$

where $\mathcal{D}$ is a numeric matrix and

$$T^{\bar{k}} = \mathrm{Tr}_{(2\ldots k)}\, R_{12}R_{23}\ldots R_{k-1,k}T_1T_2\ldots T_k \tag{2.9}$$

$$\sigma_k(T) = \alpha_k\ \mathrm{Tr}_{(12\ldots p)}\, P_-^{(p)}T_1T_2\ldots T_k. \tag{2.10}$$

In contrast to the algebra $\mathcal{L}_q(R)$, the quantities (2.10) are not central; they only form a commutative subalgebra of $\mathcal{T}_q(R)$. It is this property that prevents us from defining a quantum orbit in $\mathcal{T}_q(R)$ algebra as a quotient algebra over an ideal generated by the elements $\sigma_k(T) - c_k$ since due to the non-centrality of $\sigma_k(T)$ the corresponding quotient would not be a flat deformation of its classical counterpart.

Now we consider a special case of the quasiclassical Hecke symmetry related to the QG $U_q(sl(n))$ (see the introduction). In this case the R-matrix is a deformation of the usual permutation P : $P_{12}(v_1 \otimes v_2) = v_2 \otimes v_1$. This means that $\lim_{q\to 1} R = P$. Therefore, at the limit $q \to 1$ the quadratic quantum algebra (2.4) turns into the commutative algebra $\mathcal{L}(P) = \lim_{q\to 1} \mathcal{L}_q(R)$:

$$P_{12}L_1P_{12}L_1 - L_1P_{12}L_1P_{12} \equiv L_1L_2 - L_2L_1 = 0$$

where, as usual, $L_2 = P\,L_1\,P$. Let us pass from the quadratic algebra $\mathcal{L}_q(R)$ to a *quadratic–linear* algebra $\mathcal{L}_{q,\hbar}(R)$ with two parameters q and $\hbar$, which can be treated as a deformation of $\mathcal{L}_q(R)$. Note that this deformation is also well defined in a non-quasiclassical case. The algebra $\mathcal{L}_{q,\hbar}(R)$ will be introduced by the following simple procedure. On shifting the generators of $\mathcal{L}_q(R)$ $l_j^i = \bar{l}_j^i - h\delta_j^i$ we arrive at the equivalent algebra:

$$R_{12}\bar{L}_1R_{12}\bar{L}_1 - \bar{L}_1R_{12}\bar{L}_1R_{12} = \lambda h(R_{12}\bar{L}_1 - \bar{L}_1R_{12}).$$

Now redefining the combination λh as a new parameter $\hbar$ and treating it as *independent* of q, we obtain the two-parameter quadratic–linear algebra $\mathcal{L}_{q,\hbar}(R)$

$$R_{12}\bar{L}_1R_{12}\bar{L}_1 - \bar{L}_1R_{12}\bar{L}_1R_{12} = \hbar(R_{12}\bar{L}_1 - \bar{L}_1R_{12}). \tag{2.11}$$

We retain the notation $\bar{L}$ for the quantum matrix formed by the generators of algebra $\mathcal{L}_{q,\hbar}(R)$ in order to distinguish it from the matrix L. In the quasiclassical case the algebra $\mathcal{L}_{q,\hbar}(R)$ can be considered as a two-parameter deformation of the commutative algebra $\mathcal{L}(P) = k(gl(n)^*) = \mathrm{Sym}(gl(n))$ since, as is evident from (2.11),

$$\mathcal{L}(P) = \lim_{\substack{q\to 1\\ \hbar\to 0}} \mathcal{L}_{q,\hbar}(R) \qquad \mathcal{L}_q(R) = \lim_{\substack{\hbar\to 0\\ q=\mathrm{const}}} \mathcal{L}_{q,\hbar}(R). \tag{2.12}$$

It is important that all these deformations are flat. Otherwise stated, the Poincaré series of $\mathcal{L}(P)$, $\mathcal{L}_q(R)$ and that of the graded algebra associated with $\mathcal{L}_{q,\hbar}(R)$ are equal to each other. This statement is also valid in the non-quasiclassical case if limits (2.12) exist. Besides, the algebra $\mathcal{L}_{q,\hbar}(R)$ admits a nontrivial classical limit—the noncommutative algebra

$$\mathcal{L}_\hbar = U(gl(n)_\hbar) = \lim_{\substack{q\to 1\\ \hbar=\mathrm{const}}} \mathcal{L}_{q,\hbar}(R). \tag{2.13}$$

Indeed, as $q \to 1$ the quadratic–linear relations (2.11) turn into the following ones:

$$A_1A_2 - A_2A_1 = \hbar(A_1P_{12} - P_{12}A_1) \tag{2.14}$$

which are just relations (1.5). Moreover, we have

$$\mathcal{L}(P) = \lim_{\hbar\to 0} \mathcal{L}_\hbar.$$

Accordingly with what was said in the introduction, in the quasiclassical case we treat the algebras $\mathcal{L}_q(R)$, $\mathcal{L}_{q,\hbar}(R)$, and $U(gl(n)_\hbar)$ respectively as 'quantum commutative', 'quantum non-commutative' and 'classical non-commutative' counterparts of the commutative algebra $\mathcal{L}(P)$.

The crucial point is that for the algebras $\mathcal{L}_{q,\hbar}(R)$ and $U(gl(n)_\hbar)$ there also exists some version of the CH identity. For the matrix $\bar{L}$ formed by the generators of the algebra $\mathcal{L}_{q,\hbar}(R)$ the CH identity is similar to (2.6):

$$(-\bar{L})^p + \sum_{k=0}^{p-1}(-\bar{L})^k\sigma^{(\hbar)}_{p-k}(\bar{L}) \equiv 0 \tag{2.15}$$

(this relation is valid in a non-quasiclassical case as well). By passing to the limit $q \to 1$ we obtain a version of the CH identity for the matrix A formed by the generators of $U(gl(n)_\hbar)$:

$$(-A)^p + \sum_{k=0}^{p-1}(-A)^k\tau^{(\hbar)}_{p-k}(A) \equiv 0 \tag{2.16}$$

(see the introduction for other versions of the CH identity). The coefficients $\sigma_k^{(\hbar)}$ and $\tau_k^{(\hbar)}$ are central elements of the corresponding algebras. The explicit form of coefficients $\sigma_k^{(\hbar)}$ and the proof of the existence of their limits (denoted $\tau_k^{(\hbar)}$) as $q \to 1$ are presented in the appendix.

Let us pass now to quantum analogues of the generic orbits in question. We introduce such a 'quantum orbit' (in both the quasiclassical and non-quasiclassical cases) as the quotient of the RE algebra over the ideal generated by the elements (1.4). Keeping in mind the fact that the centre Z is generated by the elements $\sigma_k(L)$ we can define the character χ by imposing $\chi(\sigma_k(L)) = c_k$. In other words, we define the quantum orbit in question by the system of polynomial equations

$$\sigma_k(L) - c_k = 0 \qquad k = 1, \ldots, p. \tag{2.17}$$

Then the polynomial $\overline{P}$ mentioned in the introduction becomes

$$\overline{P} = (-L)^p + \sum_{k=1}^{p}(-L)^k c_{p-k}. \tag{2.18}$$

We consider the quotient of the RE algebra over the ideal generated by the lhs of (2.17) as a quantum commutative orbit. Let us denote this quotient by $k(M^q_\mu)$, $\mu = (\mu_1, \ldots, \mu_p)$, μ_i being roots of the polynomial $\overline{P}$. Let us remark that in the quasiclassical case the quotient $k(M^q_\mu)$ is a flat deformation of its classical counterpart $k(M_\mu)$, being the coordinate ring of a semisimple orbit. This can be shown by the methods of [D1]. In a similar way we can introduce the non-commutative quantum orbit $k(M^{q\hbar}_\mu)$ as the quotient of the algebra $\mathcal{L}_{q,\hbar}(R)$ over the ideal generated by $\sigma_k^{(\hbar)} - c_k$ (with a similar meaning of μ). The corresponding polynomial (2.18) can be obtained if we replace the coefficients $\sigma_k^{(\hbar)}(\bar{L})$ in (2.15) by c_k. In the quasiclassical case by passing to the limit $q \to 1$ we obtain a classical non-commutative orbit (its 'coordinate ring' will be denoted $k(M^\hbar_\mu)$) and the corresponding polynomial (2.18). In all cases we suppose the roots $\mu_1, \ldots, \mu_n$ of corresponding polynomials to be pairwise distinct.

3. Quasiclassical case: related Poisson structures

In this section we will briefly describe quasiclassical counterparts of the algebras $\mathcal{L}_q(R)$ and $\mathcal{L}_{q,\hbar}(R)$ and their restrictions to the orbits in question, assuming R to be of quasiclassical Hecke symmetry (see the introduction). Let

$$r = \sum X_\alpha \wedge X_{-\alpha} \in \wedge^2(\mathfrak{g}) \tag{3.1}$$

be the classical r-matrix related to a simple classical Lie algebra $\mathfrak{g}$. The quasiclassical counterpart of the RTT algebra is well known. It is the so-called Sklyanin bracket, defined as the difference between the left-invariant and right-invariant brackets on the corresponding Lie group G associated with r-matrix (3.1); these invariant brackets are not Poisson separately. The Sklyanin bracket can be reduced to any semisimple orbit in $\mathfrak{g}^*$. It can be quantized in the sense of deformation quantization (cf [DG]). The resulting algebra is $U_q(\mathfrak{g})$-covariant. This quantum algebra can be also treated in terms of the Hopf–Galois extension mentioned in the introduction.

However, the reduced Sklyanin bracket well defined on any semisimple orbit is not defined on the whole of $\mathfrak{g}^*$. Roughly speaking, we say that the Poisson brackets defined on each semisimple orbit separately cannot be 'glued' into a global Poisson bracket.

By contrast, the quasiclassical counterpart of the RE algebra is well defined on the whole $gl(n)^*$. Let us describe it. Let us put $\mathfrak{g} = sl(n)$ and associate with the r-matrix (3.1) a bi-vector field arising from the representation

$$ad^* : \mathfrak{g} \to \mathrm{Vect}\,(\mathfrak{g}^*).$$

By applying this bi-vector field to functions f and g we obtain a bracket $\{f, g\}_r$ which is not Poisson. Nevertheless, by adding some invariant summand we can convert it into a Poisson bracket. This summand can be constructed as follows. It is well known that in the decomposition of $\mathfrak{g}^{\otimes 2}$ into a direct sum of irreducible $\mathfrak{g}$-modules the component isomorphic to $\mathfrak{g}$ itself occurs twice: once in the symmetric part $\mathrm{Sym}^2(\mathfrak{g})$ of $\mathfrak{g}^{\otimes 2}$ and once in the skew-symmetric part $\wedge^2(\mathfrak{g})$ (as usual, we assume that $\mathfrak{g}$ acts on itself by the adjoint action and this action is extended to $\mathfrak{g}^{\otimes 2}$ via the Leibniz rule). Denote these components by $\mathfrak{g}_s$ and $\mathfrak{g}_a$ respectively:

$$\left.\begin{array}{l} \mathfrak{g}_s \subset \mathrm{Sym}^2(\mathfrak{g}) \\ \mathfrak{g}_a \subset \wedge^2(\mathfrak{g}) \end{array}\right\} \qquad \mathfrak{g}_{a,s} \sim \mathfrak{g}.$$

Let us consider a non-trivial $\mathfrak{g}$-morphism sending the component $\mathfrak{g}_a$ to $\mathfrak{g}_s$ (it is unique up to a factor). Let us extend this map to other components of $\wedge^2(\mathfrak{g})$ by zero. Let $\{f, g\}_{\mathrm{inv}}$ be the extension of this map from $\wedge^2(k(\mathfrak{g}^*))$ to $k(\mathfrak{g}^*)$ via the Leibniz rule. Then there exist two values of a such that the sum

$$\{f, g\} = \{f, g\}_r + a\{f, g\}_{\mathrm{inv}} \tag{3.2}$$

is a Poisson bracket (cf [DGS]). One of these two Poisson brackets is a quasiclassical counterpart of the RE algebra (the other one corresponds to a modified form of the RE algebra). In the following the appropriate a is assumed to be fixed. By this the corresponding Poisson bracket is defined on $sl(n)^*$ (note that it is quadratic). In order to pass to a Poisson bracket defined on $gl(n)^*$ we should add one more generator, which Poisson commutes with all other generators. Let us observe that the bracket (3.2) (or its extension to $gl(n)^*$) is compatible with the corresponding linear Poisson–Lie bracket. The Poisson pencil generated by these two brackets on $gl(n)^*$ is just the quasiclassical counterpart of the two-parameter family $\mathcal{L}_{q,\hbar}(R)$.

As for other simple Lie algebras $\mathfrak{g}$ any invariant correction to the bracket $\{\,,\,\}_r$ converting it into a quadratic Poisson bracket (cf [DGS]) does not exist.

Now let us pass to the quasiclassical counterparts of the quantum orbits above. Observe that the Poisson bracket (3.2) can be restricted to any orbit in $sl(n)^*$ (or in $gl(n)^*$ if we take its extension) (cf [D2] for a proof). In particular, this is so for semisimple generic orbits. Moreover, this restricted bracket is compatible with the Kirillov–Kostant–Souriau one. The Poisson pencil generated by these two brackets is just the quasiclassical counterpart of the algebra $k(M_\mu^{q\hbar})$ above.

Let us remark that for symmetric orbits the reduced Sklyanin bracket becomes a particular case of this Poisson pencil. Thus, for such a type of orbit the quantum objects can be described in two ways: in terms of the Hopf–Galois extension or as a quotient of the RE algebra or its quadratic–linear counterpart $\mathcal{L}_{q,\hbar}(R)$. However, the latter way is more explicit and leads to objects of 'quantum affine algebraic geometry' (cf [DGK], where the orbits of $\boldsymbol{CP}^n$ type were quantized in the spirit of such a type of geometry by means of an operator method). For non-symmetric orbits the quotients of the algebra $\mathcal{L}_q(R)$ (or $\mathcal{L}_{q,\hbar}(R)$) and algebras arising from the reduced Sklyanin bracket are completely different in spite of the fact that they are both $U_q(\boldsymbol{g})$-covariant. Also note that the family of $U_q(\boldsymbol{g})$-covariant algebras which are deformations of the coordinate ring of a semisimple orbit in $gl(n)^*$ (or $sl(n)^*$) is large enough. The reduced Sklyanin bracket or the Poisson brackets corresponding to the algebra $\mathcal{L}_q(R)$ or $\mathcal{L}_{q,\hbar}(R)$ represent only particular cases of Poisson structures corresponding to this family (cf [DGS] where such Poisson structures are classified).

4. Basic line bundles over quantum orbits

In this section we introduce quantum line bundles over the quantum orbits in question associated with the fundamental vector $gl(n)$- (or, which is the same, $sl(n)$-) module V. Let us fix an 'orbit' which is a quotient of one of the algebras $\mathcal{L}_q(R)$, $\mathcal{L}_{q,\hbar}(R)$ or $\mathcal{L}_\hbar = U(gl(n)_\hbar)$ over the ideal generated by (1.4) (in the case of the algebras $\mathcal{L}_q(R)$ and $\mathcal{L}_{q,\hbar}(R)$ in both the quasiclassical and non-quasiclassical cases).

Similarly to the above consideration we assume the roots $\mu_1, \ldots, \mu_n$ of the corresponding polynomial (2.18) to be pairwise distinct. In the following we use the notation $k(\overline{M}_\mu)$ for such an orbit (of one of the types above). Let

$$\mathcal{R}(V, \overline{M}_\mu) = V \otimes k(\overline{M}_\mu)$$

be a free right $k(\overline{M}_\mu)$-module and $\mathcal{R}_\nu$ its submodule generated by the rhs of (1.6). Let us consider the quotient module $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_\nu$.

Theorem 1. *The $k(\overline{M}_\mu)$-module $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_\nu$ is non-trivial iff ν in (1.6) coincides with one of μ_i, that is iff $\nu = \mu_i$ for some i. In this case the module $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_\nu$ is projective. More precisely,*

$$\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_{\mu_i} = \mathrm{Im}\ P_i \tag{4.1}$$

where

$$P_i = \prod_{j \neq i}^{p} \frac{L - \mu_j\,\mathrm{id}}{\mu_i - \mu_j} \tag{4.2}$$

is a projector acting on the free module $\mathcal{R}(V, \overline{M}_\mu)$. Furthermore, in the quasiclassical case the module $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_{\mu_i}$ is a flat deformation of its classical counterpart. (In the case of the algebra $U(gl(n)_\hbar)$ we assume $L = A$.)

Remark 2. Let us note that we do not consider the $k(\overline{M}_\mu)$-module $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_{\mu_i}$ as a quantum variety since the corresponding 'coordinate ring' is not well defined. In order to introduce such a ring we should define a commutation rule between the space V and the algebra $k(\overline{M}_\mu)$. However, apparently there is no reasonable way to do this (if we want to preserve the flatness of the deformation). We are planning to return to this question in a future publication.

Proof. The proof of the theorem is based on the CH identities (2.6), (2.15) and (2.16) and looks like that in the classical case because the main difficulty is hidden in the quantum version of the CH identity. The necessity of restriction $\nu = \mu_i$ for some i follows from relation (1.6) and $\overline{P}(L) = 0$ where $\overline{P}$ is defined by (2.18). Note that the latter relation can be rewritten in the form

$$\prod_{i=1}^{p}(L - \mu_i \mathrm{id}) = 0 \qquad p = \mathrm{rank}\,(R).$$

Now, by virtue of (1.6) we have

$$(\nu - \mu_1)\dots(\nu - \mu_p)v = 0 \qquad v = (v_1, \dots, v_n)$$

and if $\forall i\ \nu \neq \mu_i$ we have $v = 0$; that is, the module $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_\nu$ is trivial. In order to prove the non-triviality and projectivity of the module in the case $\nu = \mu_i$ we consider the projectors (4.2). Note that the action of the projectors P_i on the $k(\overline{M}_\mu)$-module $\mathcal{R}(V, \overline{M}_\mu)$ is given by that of the matrix L, which is defined as follows:

$$\sum_{i=1}^{n} v_i g^i(l) \triangleleft L = \sum_{i,j=1}^{n} v_j l_i^j g^i(l). \tag{4.3}$$

Thus, relation (1.6) can be represented in the form

$$v \triangleleft L - \nu v \qquad v \in V. \tag{4.4}$$

Let us remark that the action (4.3) of the matrix L on the space V is coordinated with the coaction of the Hopf algebra $\mathcal{T}_q(R)$ in general and therefore (in the quasiclassical case) with the action of the dual object, namely the QG $U_q(sl(n))$. Taking into account that μ_i are distinct we have the following. □

Proposition 1. *The operators (4.2) form the full set of orthonormal projective operators on the space $\mathcal{R}(V, \overline{M}_\mu)$, that is the following properties hold:*

(i) $P_i P_j = \delta_{ij} P_i$,
(ii) $\sum_{i=1}^{n} P_i = \mathrm{id}$.

Proof is left to the reader as an easy exercise. Let us return to the proof of the theorem. As follows from the proposition the quotient $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_{\mu_i}$ can be identified with Im P_i. This shows that the $k(\overline{M}_\mu)$-module $\mathcal{R}(V, \overline{M}_\mu)/\mathcal{R}_\nu$ is projective. Moreover, in the quasiclassical case it also implies that this module is a flat deformation of its classical counterpart, since under a deformation the projectors are deformed smoothly. This completes the proof. □

5. Derived line bundles

In this section we consider the problem of constructing the quantum line bundles different from the basic ones. We call them *derived line bundles*. First, consider the classical case. Let us fix a generic semisimple orbit M_μ and two line bundles E_{μ_i}, $i = 1, 2$ (see the introduction). Let us consider their tensor product. We want to represent its coordinate ring as a $k(M_\mu)$-module as well. This can be done as follows.

Let V again be the fundamental vector $sl(n)$-module. Consider the free right $k(M_\mu)$-module

$$\mathcal{R}(V^{\otimes 2}, M_\mu^q) = V^{\otimes 2} \otimes k(M_\mu)$$

and extend the action of the matrix $L = A$ to the space $V^{\otimes 2}$ by setting

$$(u \otimes v) \triangleleft L = u \otimes (v \triangleleft L) + (u \triangleleft L) \otimes v \qquad u, v \in V$$

(some sort of Leibniz rule). Thus, the extended matrix L, which will be denoted $L^{(2)}$, can be written as

$$L_1 + L_2 \qquad L_1 = \mathrm{id} \otimes L \qquad L_2 = P_{12} L_1 P_{12}.$$

Let us consider the submodule $\mathcal{R}_\nu$ defined as in (4.4) but with $v \in V$ replaced by $u \otimes v$. Then the same problem arises: for what value of ν is the factor

$$\mathcal{R}(V^{\otimes 2},\ M^q_\mu)/\mathcal{R}_\nu$$

not trivial? It is not difficult to see that it is so iff $\nu = \mu_i + \mu_j$ where μ_i are the roots of the polynomial $\overline{P}$, i.e. 'eingenvalues' of the matrix L corresponding to the orbit in question. This allows us to find the CH identity for the matrix $L^{(2)}$.

Also it can be found directly from the identity for the matrix L in the following way. Let $\overline{P}(L) = 0$ be the CH identity for the matrix L (in this section the coefficients of polynomial $\overline{P}$ are supposed to be numerical). Then it is not difficult to find an analogous relation for the matrix $L^{(2)}$. For this it is sufficient to raise this matrix to the powers $1, 2, \ldots, l(l+1)/2$, where l is the degree of the polynomial $\overline{P}$ and on expressing the powers $L_1^l, L_1^{l+1}, \ldots$ through $L_1^1, \ldots, L_1^{l-1}$ and similarly for the matrix L_2 we obtain the CH identity for the matrix $L^{(2)}$. The crucial property used in the construction is the mutual commutativity of the matrices L_1 and L_2:

$$L_1 L_2 = L_2 L_1.$$

Moreover, by assuming the sums $\mu_i + \mu_j$ to be pairwise distinct we can construct the projector analogous to (1.3). Nevertheless, we should restrict ourselves to the symmetric part of the space $V^{\otimes 2}$ to eliminate the multiplicity of the quantity $\mu_i + \mu_j$ since (if $i \neq j$) it occurs once in the symmetric part of this space and once in its skew-symmetric part. Thus, the projector corresponding to the eigenvalue $\mu_i + \mu_j$ is the product of the projector onto the symmetric part of $V^{\otimes 2}$ and that resembling (1.3). Let us point out that in a similar way it is possible to extend the matrix L to the higher tensor powers of the space V: $V^{\otimes l}, l = 3, 4, \ldots$. Thus, if $l = 3$ the extended matrix is defined as $L_1 + L_2 + L_3$ with the obvious definition of the matrix L_3. The details are left to the reader.

Turn now to the quantum case; i.e. assume that the matrix L is subject to relations (2.4). If we considered the matrix $L^{(2)} = L_1 + L_2$ with L_2 defined as above but with P_{12} replaced by R_{12} we would be unable to find the CH identity for such an extension of the matrix L, since the matrix L_2 does not satisfy the polynomial relation valid for $L_1 = L$. The point is that the matrices L_1 and L_2 are not similar. (However, if we chose as L_2 the matrix $R_{12} L_1 R_{12}^{-1}$ then the matrices L_1 and L_2 would become similar, but the commutativity $L_1 L_2 = L_2 L_1$ valid in the previous case by virtue of the RE would be lost.)

Nevertheless, we are interested in an extension of the matrix L to the symmetric part of the space $V^{\otimes 2}$. Let us define such an extension as follows:

$$L_+ = P_+ L P_+ \qquad \text{where} \quad P_+ = \frac{q^{-1}\mathrm{id} + R_{12}}{q + q^{-1}}. \tag{5.1}$$

Such a way to extend the matrix L to the symmetric part of $V^{\otimes 2}$ is motivated by the following observation. In the classical case ($q = 1$) such an extension of the matrix $L = A$ coincides (up to a factor, which does not matter for us) with the restriction of the matrix $L_1 + L_2$ to the symmetric part of $V^{\otimes 2}$.

In the following we will restrict ourselves to the case rank $(R) = 2$. This implies that the CH identity for the matrix L is quadratic:

$$L^2 - aL + b\,\mathrm{id} = 0 \qquad a = \mu_1 + \mu_2 \quad b = \mu_1 \mu_2. \tag{5.2}$$

Proposition 2. *If the CH identity for the matrix L is of the form (5.2) then the matrix L_+ defined by (5.1) obeys the CH identity of the form*

$$L_+^3 - a\left(1 + \frac{q^{-1}}{2_q}\right) L_+^2 + \left(a^2\frac{q^{-1}}{2_q} - b\right) L_+ + ab\frac{q^{-1}}{2_q}\,\mathrm{id} = 0. \tag{5.3}$$

Proof. Taking into account the formulas for the symmetrizer $P_+^{(2)}$ we can express the R-matrix via P_+ and rewrite the RE algebra in the equivalent form:

$$P_+LP_+L - LP_+LP_+ + \frac{q^{-1}}{2_q}(L^2P_+ - P_+L^2) = 0. \tag{5.4}$$

If the matrix L obeys the CH identity (5.2) we then have

$$P_+LP_+L - LP_+LP_+ + a\frac{q^{-1}}{2_q}(LP_+ - P_+L) = 0. \tag{5.5}$$

Now the CH identity for L_+ is a consequence of direct calculations. Indeed, let us calculate successively the powers of matrix L_+. Below we use the abbreviation $\xi = a\frac{q^{-1}}{2_q}$. For L_+^2 we obtain

$$L_+^2 \equiv (P_+LP_+L)P_+ = (\text{use (5.5)}) = LL_+ - \xi LP_+ + \xi L_+.$$

We have here unwanted terms LL_+ and LP_+ and therefore should calculate the next power of L_+ in order to get rid of them. So

$$\begin{aligned} L_+^3 = L_+^2L_+ &= (\text{insert } L_+^2 \text{ above}) = LL_+^2 - \xi LL_+ + \xi L_+^2 \\ &= (\text{insert } L_+^2 \text{ again and use (5.2)}) \\ &= a\left(1 + \frac{q^{-1}}{2_q}\right) LL_+ - a\left(1 + \frac{q^{-1}}{2_q}\right)\xi LP_+ + (\xi^2 - b)L_+ + b\xi P_+. \end{aligned}$$

No new unwanted term besides LP_+ and LL_+ appears and excluding these from expressions for L_+^2 and L_+^3 we arrive at the following:

$$L_+^3 - a\left(1 + \frac{q^{-1}}{2_q}\right) L_+^2 + \left(a^2\frac{q^{-1}}{2_q} + b\right) L_+ - ab\frac{q^{-1}}{2_q}P_+ = 0.$$

It remains to observe that $P_+ = \mathrm{id}$ on the symmetric part of $V^{\otimes 2}$. In a similar way we can extend the matrix L to the higher symmetric powers of the space V: it suffices to replace the projectors P_+ in the formula (5.1) by the symmetrizer in a given power. However, the problem of finding the corresponding CH identity is much more complicated and is still open. □

Acknowledgments

The authors are supported by the grant PICS-608/RFBR 98-01-22033. One of the authors (PS) acknowledges the Laboratoire de Mathématiques de l'Université de Valenciennes for hospitality; during his stay there this paper was started.

Appendix

In this appendix we present the explicit form of the coefficients entering (2.15) and show that in the quasiclassical case these coefficients have finite limit as $q \to 1$. This fact gives rise to the formula (2.16). Let us recall that in general $p = \mathrm{rank}(R) \neq n = \dim V$. Also recall that the degree of the polynomial $\overline{P}$ and hence the number of the roots μ_i is equal to p. We will use the fact that the $\mathcal{L}_{q,\hbar}(R)$ algebra (2.11) can be formally obtained from $\mathcal{L}_q(R)$ by the shift of generators $l^i_j = \bar{l}^i_j - \delta^i_j h$ and subsequent changing of the parameter $\hbar = h\lambda$. So, if we realize these two operations in the CH identity (2.6) for the matrix L we obtain some polynomial identity on the matrix $\bar{L}$ formed by the generators of the algebra $\mathcal{L}_{q,\hbar}(R)$.

The problem is to find in which way the central elements $\sigma_k(L)$ are transformed. This can be directly calculated from the definition of $\sigma_k(L)$ (2.7). Indeed, on making the above-mentioned shift in $\sigma_k(L)$ we have to transform the number of arising terms, their typical form being

$$\mathrm{Tr}_{(12\ldots p)}\, P_-^{(p)}(L_1R_1\ldots R_{k-1})^s R_1\ldots R_{k-1}(L_1R_1\ldots R_{k-1})^{k-s-1}. \tag{A.1}$$

In the above formula the concise notation $R_i \equiv R_{i\,i+1}$ is used. Now one should 'draw out' the string of R-matrices and cancel them on the projector $P_-^{(p)}$. Then it is necessary to get rid of all the matrices R_{k-1} in monomials $(L_1R_1\ldots R_{k-1})$. The basic formulas for such transformations are

$$(\bar{L}_1R_1\ldots R_k)R_i = R_{i+1}(\bar{L}_1R_1\ldots R_k) \qquad \forall\, i \leqslant k-1$$

(this is a trivial consequence of the YB equation) and

$$(\bar{L}_1R_1\ldots R_k)^r = (\bar{L}_1R_1\ldots R_{k-1})^r R_kR_{k-1}\ldots R_{k-r+1}$$

which in turn is a direct consequence of the previous relation. The cancellation of R-matrices in (A.1) is due to the cyclic property of the trace and the defining property of the skew-symmetrizer $P_-^{(r)}$:

$$R_i\, P_-^{(r)} = P_-^{(r)}\, R_i = -\frac{1}{q}\, P_-^{(r)} \qquad \forall i \leqslant r-1.$$

Now after straightforward calculations we arrive at the following transformation of coefficients:

$$\sigma_k(L) \longrightarrow \sum_{r=0}^{k}\left(-\frac{\hbar}{\lambda}\right)^r q^{-r(p-1)} C_k^p\, \frac{[C_r^k]_q}{[C_p^{k-r}]_q}\,\sigma_{k-r}(\bar{L}). \tag{A.2}$$

Here the symbol $[C_p^k]_q$ represents the q-binomial coefficient

$$[C_p^k]_q = \frac{p_q!}{k_q!(p-k)_q!}$$

where $p_q! = (p-1)_q!p_q$ and q-numbers r_q are defined as

$$r_q \equiv \frac{q^r - q^{-r}}{q - q^{-1}}.$$

The elements $\sigma_k(\bar{L})$ are defined in (2.7), where matrix L should be changed for $\bar{L}$. Obviously, these elements are central in the $\mathcal{L}_{q,\hbar}(R)$ algebra. Now given the rule (A.2) it is not too difficult to obtain from (2.6) the CH identity for the $\mathcal{L}_{q,\hbar}(R)$ algebra

$$(-\bar{L})^p + \sum_{k=0}^{p-1}(-\bar{L})^k\sigma^{(\hbar)}_{p-k}(\bar{L}) \equiv 0. \tag{2.15}$$

The quantities $\sigma^{(\hbar)}_{p-k}(\bar{L})$ are the following polynomials in $\hbar$ with coefficients depending on $\sigma_{p-k}(\bar{L})$:

$$\sigma^{(\hbar)}_{p-k}(\bar{L}) = \sigma_{p-k}(\bar{L}) + \sum_{r=1}^{p-k} \hbar^r \omega^{(p)}_{r+k,k}\, \sigma_{p-k-r}(\bar{L}). \tag{A.3}$$

The numeric coefficients $\omega^{(p)}_{s,k}$ are as follows ($s > k$):

$$\omega^{(p)}_{s,k} = \frac{\lambda^{k-s}}{[C^s_p]_q} \sum_{r=0}^{s-k} (-1)^r q^{-r(p-1)} C^k_{s-r} C^r_{p-s+r} [C^{s-r}_p]_q. \tag{A.4}$$

Now we calculate the sum in (A.4) and show that it is proportional to λ^{s-k} and therefore the whole coefficient $\hbar^r \omega^{(p)}_{r-k,k}$ admits a non-singular classical limit as $q \to 1,\ \hbar = \text{const}$. On taking this limit in the CH identity (2.15) we obtain the corresponding identity (2.16) for the matrix A of the $U(gl(n)_\hbar)$ algebra (2.14). Denote the sum in (A.4) by $\xi^{(p)}_{s,k}$:

$$\xi^{(p)}_{s,k} \equiv \sum_{r=0}^{s-k} (-1)^r q^{-r(p-1)} C^k_{s-r} C^r_{p-s+r} [C^{s-r}_p]_q$$

and calculate the generating function $\Phi_\xi(x, y)$ of the coefficients $\xi^{(p)}_{s,k}$

$$\Phi_\xi(x, y) \stackrel{\text{def}}{=} \sum_{s=0}^{p} \sum_{k=0}^{s} (-x)^{p-s} (-y)^k \xi^{(p)}_{s,k}.$$

If one knew the function $\Phi_\xi(x, y)$ then the coefficients $\xi^{(p)}_{s,k}$ could be found as

$$\xi^{(p)}_{s,k} = \frac{(-1)^{p-s+k}}{k!(p-s)!} \left[\frac{\partial^{p-s}}{\partial x^{p-s}} \frac{\partial^k}{\partial y^k} \Phi_\xi(x, y) \right]_{x=y=0}. \tag{A.5}$$

The calculation of $\Phi_\xi(x, y)$ is rather simple where the only thing we need is the Newton binomial formula and its q-analogue

$$\sum_{k=0}^{p} x^k q^{-k(p-1)} [C^k_p]_q = \prod_{k=0}^{p-1} \left(1 + \frac{x}{q^{2k}} \right).$$

So we present the final result

$$\Phi_\xi(x, y) = (-1)^p q^{-p(p-1)} \prod_{k=0}^{p-1} (xq^{p-1} + yq^{2k} - \lambda\, q^k k_q). \tag{A.6}$$

Finally, upon taking the partial derivatives in (A.5) we find the form of coefficients $\xi^{(p)}_{s,k}$:

$$\xi^{(p)}_{s,k} = \lambda^{s-k} q^{\frac{(p-1)(p-2s)}{2}} (p-1)_q!\, (V_k + V_s), \tag{A.7}$$

where by V_k and V_s we denote the following sums:

$$V_k \equiv \begin{cases} 0 & \text{if} \quad k = 0 \\ \displaystyle\sum_{\substack{1 \leqslant l_1 < \cdots < l_{k-1} \leqslant p-1 \\ 1 \leqslant r_1 < \cdots < r_{p-s} \leqslant p-1 \\ \{l_i\} \cap \{r_j\} = \emptyset}} \frac{q^{(l_1 + \cdots + l_{k-1} - r_1 - \cdots - r_{p-s})}}{(l_1)_q \ldots (l_{k-1})_q (r_1)_q \ldots (r_{p-s})_q} & \text{if} \quad k \neq 0 \end{cases}$$

$$V_s \equiv \begin{cases} 0 & \text{if} \quad s = p \\ \displaystyle\sum_{\substack{1 \leqslant l_1 < \cdots < l_k \leqslant p-1 \\ 1 \leqslant r_1 < \cdots < r_{p-s-1} \leqslant p-1 \\ \{l_i\} \cap \{r_j\} = \emptyset}} \frac{q^{(l_1 + \cdots + l_k - r_1 - \cdots - r_{p-s-1})}}{(l_1)_q \ldots (l_k)_q (r_1)_q \ldots (r_{p-s-1})_q} & \text{if} \quad s \neq p. \end{cases}$$

In particular,

$$\xi^{(p)}_{p,p} \equiv 1 \qquad \xi^{(p)}_{p,0} \equiv 0.$$

Now we can find the coefficients $\omega^{(p)}_{s,k}$ entering the CH identity (2.15) of the $\mathcal{L}_{q,\hbar}(R)$ algebra

$$\omega^{(p)}_{s,k} \equiv \frac{\lambda^{k-s}}{[C^s_p]_q}\xi^{(p)}_{s,k} = q^{\frac{(p-1)(p-2s)}{2}}\frac{(p-1)_q!}{[C^s_p]_q}(V_k + V_s).$$

This gives the explicit form of $\sigma^{(\hbar)}_k$ and completes the proof of the CH identity for the algebra $\mathcal{L}_{q,\hbar}(R)$. From the above relation it is obvious that the coefficients $\omega^{(p)}_{s,k}$ admit a non-singular classical limit

$$\lim_{q\to 1}\omega^{(p)}_{s,k} \equiv \rho^{(p)}_{s,k} = \frac{(p-1)!}{C^s_p}\left(V^{cl}_k + V^{cl}_s\right) \tag{A.8}$$

where $V^{cl}_{k,s}$ are given by formulas for $V_{k,s}$ with substitution $q = 1$ and all q-numbers changed for ordinary ones.

Finally, on taking into account that $U(gl(n)_\hbar) = \lim_{q\to 1}\mathcal{L}_{q,\hbar}(R)$ we deduce from (2.15) the CH identity (2.16), where coefficients $\tau^{(\hbar)}_{p-k}(A)$ are the classical limit of $\sigma^{(\hbar)}_{p-k}(\bar{L})$ in (A.3):

$$\tau^{(\hbar)}_{p-k}(A) = \lim_{q\to 1}\sigma^{(\hbar)}_{p-k}(\bar{L}) = \tau_{p-k}(A) + \sum_{s=1}^{p-k}\hbar^s\rho^{(p)}_{s+k,k}\,\tau_{p-k-s}(A) \tag{A.9}$$

with $\rho^{(p)}_{s,k}$ defined in (A.8). The central elements $\tau_k(A)$ have the form

$$\tau_k(A) = \frac{C^k_p}{p!}\,\varepsilon_{i_1\ldots i_k a_{k+1}\ldots a_p}A^{i_1}_{j_1}\ldots A^{i_k}_{j_k}\varepsilon^{j_1\ldots j_k a_{k+1}\ldots a_p}$$

ε being the skew-symmetric Levi–Civita tensor. These elements are analogues of spectral invariants of the usual matrix with commutative entries: in that case each τ_k is the sum of all principal minors of kth order.

From identity (A.9) one can see that in the classical non-commutative case each coefficient in the CH identity is modified by adding a polynomial in $\hbar$. In particular, the free term of the identity, which can be treated as a non-commutative analogue of the determinant, is given by the following formula:

$$\det A + \sum_{k=1}^{p-1}\hbar^k\rho^{(p)}_{k,0}\tau_{p-k}(A). \tag{A.10}$$

In this formula it is taken into account that $\rho^{(p)}_{p,0} = 0$.

References

[AG] Akueson P and Gurevich D 1999 Some aspects of braided geometry: differential calculus, tangent space, gauge theory *J. Phys. A: Math. Gen.* **32** 4183–97

[BL] Biedenharn L C and Louck J D 1968 A pattern calculus for tensor operators in the unitary groups *Commun. Math. Phys.* **8** 89–131

Biedenharn L C and Louck J D 1970 Canonical unit adjoint tensor operators in $U(n)$ *J. Math. Phys.* **11** 2368–414

[D1] Donin J 1997 Double quantization on the coadjoint representation of $sl(n)$ *Czech. J. Phys.* **47** 1115–22

[D2] Donin J 1999 Double quantization on the coadjoint representations of simple Lie groups and their orbits *Preprint* MPIM-Bonn 99–103

[DG] Donin J and Gurevich D 1995 Some Poisson structures associated to Drinfeld–Jimbo R-matrices and their quantization *Israel Math. J.* **92** 23–32

[DGK] Donin J, Gurevich D and Khoroshkin S 1998 Double quantization of CP^n type orbits by generalized Verma modules *J. Geom. Phys.* **28** 384–406

[DGS] Donin J, Gurevich D and Shnider S 1999 Double quantization on some orbits in the coadjoint representations of simple Lie groups *Commun. Math. Phys.* **204** 39–60

[FRT] Faddeev L D, Reshetikhin N Yu and Takhtajan L A 1990 Quantization of Lie groups and Lie algebras *Leningrad Math. J.* **1** 193–226

[G-T] Gelfand I M, Krob D, Lascoux A, Leclerc B, Retakh V S and Thibon J-Y 1995 Noncommutative symmetric functions *Adv. Math.* **112** 218–348

[G] Gurevich D 1991 Algebraic aspects of the quantum Yang–Baxter equation *Leningrad Math. J.* **2** 801–28

[GR] Gurevich D and Rubtsov V 1995 Quantization of Poisson pencils and generalized Lie algebras *Teor. Mat. Phys.* **103** 476–88

[GPS] Pyatov P N and Saponov P A 1995 *J. Phys. A: Math. Gen.* **28** 4415–21
Gurevich D I, Pyatov P N and Saponov P A 1997 Hecke symmetries and characteristic relations on reflection equation algebras *Lett. Math. Phys.* **41** 255–64

[HM] Hajac P and Majid S 1999 Projective module description of the Q-monopole *Commun. Math. Phys.* **206** 247–64

[IOP] Isaev A, Ogievetsky O and Pyatov P 1998 Generalized Cayley–Hamilton–Newton identities *Czech. J. Phys.* **48** 1369–74

[KSk] Kulish P P and Sklyanin E K 1992 Algebraic structures related to reflection equation *J. Phys. A: Math. Gen.* **25** 5663–975

[KSa] Kulish P P and Sasaki R 1993 Covariance properties of reflection equation algebras *Prog. Theor. Phys.* **89** 741–61

[M1] Majid Sh 1991 Examples of braided groups and braided matrices *J. Math. Phys.* **32** 3246–52

[M2] Majid Sh 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)

[Sh] Schneider S 1994 Hopf Galois extensions, crossed products, and Clifford theory *Advances in Hopf Algebras (Lecture Notes in Pure and Applied Mathematics vol 158)* ed J Bergen and S Montgomery (New York: Dekker) pp 267–97

[Se] Serre J-P 1957/1958 *Modules projectifs et espaces fibrés a fibre vectorielle* Séminaire Dubreil-Pisot Fasc. 2, Exposé 23

[Sw] Swan R 1962 Vector bundles and projective modules *Trans. Am. Math. Soc.* **105** 264–77